JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 21, 445-457 (1968)

Duhamel's Principle in the Nonstationary Radiation Field

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1. INTRODUCTION

Duhamel's principle for the stationary internal radiation field in an inhomogeneous atmosphere of finite or semi-infinite optical thickness is formulated by Matsumoto [1], based on the existence and uniqueness theorem for the solution of Milne's integral equation. By this principle, the nonclassical radiation field due to arbitrary incident radiation is expressed by an integration of the radiation field due to incident radiation in a specified direction.

The stationary nonclassical radiation field at optical depth τ in direction $\cos^{-1}\mu$, which satisfies the equation of transfer in the form

$$\mu \frac{\partial}{\partial\tau} I(\tau, \mu) = I(\tau, \mu) - \Im(\tau), \tag{1}$$

where

$$\Im(\tau) = \frac{\lambda(\tau)}{2} \int_{-1}^{1} I(\tau, \mu')\, d\mu',$$

together with boundary conditions

$$\left.\begin{aligned} I(\tau_0, -\mu) &= I_{\text{inc}}(\mu) \qquad (0 < \mu \leqslant 1), \\ I(\tau_1, +\mu) &= I^*_{\text{inc}}(\mu) \qquad (0 < \mu \leqslant 1), \end{aligned}\right\} \tag{2}$$

is expressed by

$$I(\tau, \mu) = \int_0^1 I_0(\tau, \mu; \mu') I_{\text{inc}}(\mu')\, d\mu' + \int_0^1 I_1(\tau, \mu; \mu') I^*_{\text{inc}}(\mu')\, d\mu'. \tag{3}$$

In the above equations $\lambda(\tau)$, τ_0 and τ_1 are respectively the albedo for single scattering $(0 \leqslant \lambda(\tau) \leqslant 1)$, the optical depths of the upper and lower surfaces. Furthermore we denote the intensity directed towards the upper surface by $I(\tau, +\mu)(0 < \mu \leqslant 1)$, and that directed towards the lower surface by

$I(\tau, -\mu)(0 < \mu \leqslant 1)$. In Eq. (3), $I_0(\tau, \mu)$ and $I_1(\tau, \mu)$ are respectively the solutions of Eq. (1), subject to the boundary conditions

$$I_0(\tau_0, -\mu; \mu') = \delta(\mu - \mu'), I_0(\tau_1, +\mu; \mu') = 0, \tag{4}$$

$$I_1(\tau_0, -\mu; \mu') = 0, I_1(\tau_1, +\mu; \mu') = \delta(\mu - \mu'). \tag{5}$$

In the present paper, we discuss Duhamel's principle for time-dependent radiation field in an inhomogeneous atmosphere of finite or semi-infinite thickness and derive an equation similar to Eq. (3). We assume the medium in the atmosphere is independent on time, but the incident radiation is nonstationary.

2. Fundamental Equations

We consider a finite, plane-parallel, nonemitting and isotropically scattering atmosphere of arbitrary stratification. Let radiation of intensities $I_{\text{inc}}(t, \mu)$ and $I^*_{\text{inc}}(t, \mu)$ fall on the upper and lower surfaces of geometrical depths z_0 and z_1, respectively. The nonclassical radiation field, at depth z at time $t \geqslant 0$ in direction $\cos^{-1} \mu$, satisfies the time-dependent equation of transfer in the form

$$-\frac{1}{c}\frac{\partial}{\partial t} I(t, z, \mu) + \mu \frac{\partial}{\partial z} I(t, z, \mu) = \ell(z) I(t, z, \mu) - \sigma(z) J(t, z)$$
$$(0 < |\mu| \leqslant 1, \qquad t \geqslant 0), \tag{6}$$

together with the initial and boundary conditions

$$I(0, z, \mu) = 0 \; (0 < |\mu| \leqslant 1), \tag{7}$$

$$I(t, z_0, -\mu) = I_{\text{inc}}(t, \mu) \qquad (0 < \mu \leqslant 1, \quad 0 \leqslant t), \tag{8}$$

$$I(t, z_1, +\mu) = I^*_{\text{inc}}(t, \mu) \qquad (0 < \mu \leqslant 1, \quad 0 \leqslant t), \tag{9}$$

where c, $\ell(z)$ and $\sigma(z)$ are respectively the velocity of light, the monochromatic volume attenuation, and scattering coefficients such that

$$0 < \ell \leqslant \ell(z), \qquad 0 \leqslant \sigma(z) \leqslant \sigma, \qquad 0 \leqslant \frac{\sigma}{\ell} \leqslant 1. \tag{10}$$

In Eq. (6) the source function $J(t, z)$ is given by

$$J(t, z) = \frac{1}{2}\int_{-1}^{1} I(t, z, \mu')\, d\mu'.$$

The Laplace transformation of Eq. (6) together with initial condition (7) reads

$$\mu \frac{\partial}{\partial z} I(s, z, \mu) = \left[\ell(z) + \frac{s}{c}\right] I(s, z, \mu) - \sigma(z) J(s, z), \tag{11}$$

where

$$I(s, z, \mu) = \int_0^\infty I(t, z, \mu) e^{-st}\, dt, \tag{12}$$

and

$$J(s, z) = \frac{1}{2}\int_{-1}^{1} I(s, z, \mu')\, d\mu'. \tag{13}$$

We assume $I(t, z, \mu)$ is continuous and hence bounded for all $t \geq 0$. This assumption is physically permissible. Then the Laplace transform (12) is absolutely convergent for all half plane (See Widder [2])

$$\operatorname{Re} s > 0, \tag{14}$$

where Re s represents the real part of complex number s.

The Laplace transformations of boundary conditions (8) and (9) give

$$I(s, z_0, -\mu) = I_{\text{inc}}(s, \mu)\ (0 < \mu \leqslant 1), \tag{15}$$

and

$$I(s, z_1, +\mu) = I^*_{\text{inc}}(s, \mu) \qquad (0 < \mu \leqslant 1). \tag{16}$$

Treating $J(s, z)$ as known, the formal solution of equation (11) subject to conditions (15) and (16) is

$$\begin{aligned} I(s, z, +\mu) &= I^*_{\text{inc}}(s, \mu) \exp[-\{q(s, z_1) - q(s, z)\}/\mu] \\ &\quad + \int_z^{z_1} \sigma(z') J(s, z') \exp[-\{q(s, z') - q(s, z)\}/\mu] \frac{dz'}{\mu}, \end{aligned} \tag{17}$$

$$\begin{aligned} I(s, z, -\mu) &= I_{\text{inc}}(s, \mu) \exp[-\{q(s, z) - q(s, z_0)\}/\mu] \\ &\quad + \int_{z_0}^{z} \sigma(z') J(s, z') \exp[-\{q(s, z) - q(s, z')\}/\mu] \frac{dz'}{\mu}, \end{aligned} \tag{18}$$

where

$$q(s, z) = \int_0^z \ell(z')\, dz' + sz/c. \tag{19}$$

On writing Eq. (13) as

$$J(s, z) = \frac{1}{2}\int_0^1 I(s, z, +\mu')\, d\mu' + \frac{1}{2}\int_0^1 I(s, z, -\mu')\, d\mu', \tag{20}$$

substituting from Eqs. (17) and (18), we get

$$(1 - L)_z\{J(s, z')\} = B(s, z), \tag{21}$$

where linear operator L_z is defined by

$$L_z\{f(s, z')\} = \frac{1}{2}\int_{z_0}^{z_1} \sigma(z')K(s, z, z')f(s, z')\, dz', \tag{22}$$

$$K(s, z, z') = \int_0^1 \exp\left[-\left\{\left|\int_{z'}^{z} \ell(z'')\, dz''\right| + s|\, z - z'\,|/c\right\}\Big/p\right]\frac{dp}{p}$$
$$= K(s, z', z), \tag{23}$$

and

$$B(s, z) = \frac{1}{2}\int_0^1 I_{\text{inc}}(s, u')\exp[-\{q(s, z) - q(s, z_0)\}/\mu']\, d\mu'$$
$$+ \frac{1}{2}\int_2^1 I^*_{\text{inc}}(s, \mu')\exp[-\{q(s, z_1) - q(s, z)\}/\mu']\, d\mu'. \tag{24}$$

When we consider a semi-infinite atmosphere, we must put $z_1 = \infty$ and

$$I^*_{\text{inc}}(t, \mu) = 0.$$

3. The Solution of Modified Milne's Integral Equation

We call Eq. (22) modified Milne's integral equation.

Lemma 1. *Putting*

$$\rho = |\, L_z\{1\}|\,,$$

we have

$$0 \leqslant \rho < 1, \tag{25}$$

for $0 \leqslant z_0 \leqslant z \leqslant z_1 \leqslant \infty$, *subject to condition* (10).

Proof: By Eq. (22),

$$\rho = \frac{1}{2}\left|\int_{z_0}^{z_1} \sigma(z')K(s, z, z')\, dz'\right| \leqslant \frac{\sigma}{2}\int_{z_0}^{z_1} |\, K(s, z, z')|\, dz', \tag{26}$$

where condition (10) is used. Denoting the imaginary part of complex number s by Im s, from Eq. (23) we have

$$|\, K(s, z, z')| \leqslant \int_0^1 \left|\exp\left[-\left|\int_{z'}^{z} \ell(z'')\, dz''\right|\Big/p\right]\right|\, |\exp[-s|\, z - z'\,|/cp]|\, \frac{dp}{p}$$
$$= \int_0^1 \exp\left[-\left|\int_{z'}^{z} \ell(z'')\, dz''\right|\Big/p\right]\exp[-\operatorname{Re} s \cdot |\, z - z'\,|/cp]$$

$$\times \mid \exp[-i\,\mathrm{Im}\, s \cdot \mid z - z' \mid /cp] \mid \frac{dp}{p}$$

$$= \int_0^1 \exp\left[-\left\{\left|\int_{z'}^{z} \ell(z'')\,dz''\right| + \mathrm{Re}\, s \cdot \mid z - z' \mid /c\right\}/p\right] \frac{dp}{p}$$

$$= E_1\left[\left|\int_{z'}^{z} \ell(z'')\,dz''\right| + \mathrm{Re}\, s \mid z - z' \mid /c\right], \tag{27}$$

where $E_n(x)$ is the exponential integral function

$$E_n(x) = \int_1^\infty \exp(-px)p^{-n}\,dp, \qquad n = 1, 2, \ldots. \tag{28}$$

Since $E_1(x)$ is steadly deceasing, by (10) and (27),

$$\mid K(s, z, z') \mid \leqslant E_1[(\ell + \mathrm{Re}\, s/c) \mid z - z' \mid]. \tag{29}$$

Therefore we have

$$\begin{aligned} \int_{z_0}^{z_1} \mid K(s, z, z') \mid dz' &\leqslant \int_{z_0}^{z_1} E_1[(\ell + \mathrm{Re}\, s/c) \mid z - z' \mid]\, dz' \\ &\leqslant \int_0^\infty E_1[(\ell + \mathrm{Re}\, s/c) \mid z - z' \mid]\, dz' \\ &= (\ell + \mathrm{Re}\, s/c)^{-1}[2 - E_2\{(\ell + \mathrm{Re}\, s/c)z\}] \\ &\leqslant 2(\ell + \mathrm{Re}\, s/c)^{-1}. \end{aligned} \tag{30}$$

Recalling (10) and (14), from (26) and (30) we get

$$\rho \leqslant (\sigma/2)\, 2(\ell + \mathrm{Re}\, s/c)^{-1} = \sigma(\ell + \mathrm{Re}\, s/c)^{-1} < 1.$$

This completes the proof.

LEMMA 2. *If a function $f(s, z)$ is bounded by a non-negative constant K for $z_0 \leqslant z \leqslant z_1 \leqslant \infty$ and Re $s > 0$, then we have*

$$\mid L_z^\nu\{f(s, z')\} \mid \leqslant K\rho^\nu \ (\nu = 0, 1, 2, \ldots), \tag{31}$$

where L_z^ν denotes the operator L_z repeated ν times and L_z^0 is the identity operator.

PROOF: We prove this lemma by induction. For $\nu = 0$, (31) is evident. Now we assume (31) is proved for $\nu = n$, i.e.,

$$\mid L_z^n\{f(s, z')\} \mid \leqslant K\rho^n, \tag{32}$$

By Lemma 1, (32) and (22), we have

$$\begin{aligned} \mid L_z^{n+1}\{f(s, z')\} \mid &\leqslant \frac{1}{2}\left|\int_{z_0}^{z_1} K(s, z, z')\sigma(z') \mid L_{z'}^n\{f(s, z'')\} \mid dz'\right| \\ &\leqslant \mid L_z\{K\rho^n\} \mid \leqslant K\rho^n \mid L_z\{1\} \mid \leqslant K\rho^{n+1}. \end{aligned}$$

This completes the proof.

LEMMA 3. *If* $B(s, z) = 0$ *in* Eq. (21), *then there is no nonzero solution of Eq.* (21).

PROOF: We assume a solution exists and it is nonzero in a set of positive measure. By (21), we have

$$| J(s, z)| = \frac{1}{2} \left| \int_{z_0}^{z_1} \sigma(z')K(s, z, z')J(s, z')\, dz' \right|. \tag{33}$$

Integrating both sides of this equation with respect to z over (z_0, z_1), we have

$$\int_{z_0}^{z_1} | J(s, z)|\, dz \leqslant \frac{1}{2} \int_{z_0}^{z_1} | J(s, z')|\, dz' \left| \int_{z_0}^{z_1} \sigma(z')K(s, z, z')\, dz \right|.$$

By (10) and (30), we get

$$\int_{z_0}^{z_1} | J(s, z)|\, dz \leqslant \sigma(\ell + \operatorname{Re} s/c)^{-1} \int_{z_0}^{z_1} | J(s, z')|\, dz'.$$

That is

$$\{1 - \sigma(\ell + \operatorname{Re} s/c)^{-1}\} \int_{z_0}^{z_1} | J(s, z)|\, dz \leqslant 0.$$

By the assumption, this is impossible. This completes the proof.

THEOREM 1. *If* $B(s, z)$ *is bounded for* $0 \leqslant z_0 \leqslant z \leqslant z_1 \leqslant \infty$ *and* Re $s > 0$, *then there is the unique solution of integral equation* (21) *bounded for* $0 \leqslant z_0 \leqslant z \leqslant z_1 \leqslant \infty$ *and Re* $s > 0$.

PROOF: We consider the Neumann series $\sum_{\nu=0}^{\infty} L_z^{\nu}\{B(s, z')\}$. By the assumption of this theorem, there is a non-negative constant K for $0 \leqslant z_0 \leqslant z \leqslant z_1 \leqslant \infty$ and Re $s > 0$ such that

$$| B(s, z)| \leqslant K.$$

By Lemma 2, we have

$$\left| \sum_{\nu=0}^{\infty} L_z^{\nu}\{B(s, z')\} \right| \leqslant \sum_{\nu=0}^{\infty} | L_z^{\nu}\{B(s, z')\}| \leqslant K \sum_{\nu=0}^{\infty} \rho^{\nu} = K(1 - \rho)^{-1}. \tag{34}$$

Hence the Neumann series is uniformly convergent for $z_0 \leqslant z \leqslant z_1 \leqslant \infty$ and Re $s > 0$, whence

$$\sum_{\nu=0}^{\infty} L_z^{\nu}\{B(s, z')\} = \Phi(s, z). \tag{35}$$

By (34), $\Phi(s, z)$ is bounded for $z_0 \leqslant z \leqslant z_1 \leqslant \infty$ and Re $s > 0$. Operating L_z on both sides of Eq. (35),

$$\sum_{\nu=1}^{\infty} L_z^{\nu}\{B(s, z')\} = L_z\{\Phi(s, z')\},$$

i.e.,

$$\sum_{\nu=0}^{\prime} L_z^{\nu}\{B(s, z')\} - B(s, z) = L_z\{\Phi(s, z')\}.$$

Hence

$$(1 - L)_z\{\Phi(s, z')\} = B(s, z).$$

Therefore $\Phi(s, z)$ is a solution of Eq. (21). If there are two solutions $\Phi_1(s, z)$ and $\Phi_2(s, z)$, putting $\Phi^*(s, z) = \Phi_1(s, z) - \Phi_2(s, z)$, $\Phi^*(s, z)$ is a solution of Eq. (21) with $B(s, z) = 0$. Hence by Lemma 3, $\Phi^*(s, z)$ is identically zero. This completes the proof.

4. Duhamel's Principle

We consider two solutions of equation of transfer (6), $I_0(t, z, \mu; \mu')$ and $I_1(t, z, \mu; \mu')$, together with the initial and boundary conditions

$$I_0(0, z, \mu; \mu') = I_1(0, z, \mu; \mu') = 0 \ (0 < |\mu| \leqslant 1), \tag{36}$$

$$\begin{aligned} I_0(t, z_0, -\mu; \mu') &= I_1(t, z_1, +\mu; \mu') \\ &= F(t)\, \delta(\mu - \mu') \ (0 < \mu, \mu' \leqslant 1), \end{aligned} \tag{37}$$

$$\begin{aligned} I_0(t, z_1, +\mu; \mu') &= I_1(t, z_0, -\mu; \mu') = 0 \\ &\qquad (0 < \mu, \mu' \leqslant 1), \end{aligned} \tag{38}$$

where $\delta(x)$ is Dirac's δ-function and $F(t)$ will be given later. The source functions are

$$J_0(t, z; \mu') = \frac{1}{2}\int_{-1}^{1} I_0(t, z, \mu''; \mu')\, d\mu'', \tag{32}$$

and

$$J_1(t, z; \mu') = \frac{1}{2}\int_{-1}^{1} I_1(t, z, \mu''; \mu')\, d\mu''. \tag{40}$$

In what follows, we consider two cases: that in which $F(t) = \delta(t)$ (Case 1), and that in which $F(t) = H(t)$ (Case 2), where $H(t)$ is Heaviside's unit step function defined by

$$\begin{aligned} H(t) &= 0 \quad t < 0, \\ &= 1 \quad t > 0. \end{aligned} \tag{41}$$

4.1 CASE 1. $F(t) = \delta(t)$

Corresponding to Eq. (17) and (18), we have

$$I_0(s, z, +\mu; \mu') = \int_z^{z_1} \sigma(z')\bar{J}_0(s, z'; \mu') \exp[-\{q(s, z') - q(s, z)\}/\mu] \frac{dz'}{\mu}, \quad (42)$$

$$\begin{aligned} I_0(s, z, -\mu; \mu') &= \delta(\mu - \mu') \exp[-\{q(s, z) - q(s, z_0)\}/\mu] \\ &\quad + \int_{z_0}^{z} \sigma(z')\bar{J}_0(s, z'; \mu') \exp[-\{q(s, z) - q(s, z')\}/\mu] \frac{dz'}{\mu}, \end{aligned} \quad (43)$$

$$\begin{aligned} I_1(s, z, +\mu; \mu') &= \delta(\mu - \mu') \exp[-\{q(s, z_1) - q(s, z)\}/\mu] \\ &\quad + \int_{z}^{z_1} \sigma(z')\bar{J}_1(s, z'; \mu') \exp[-\{q(s, z') - q(s, z)\}/\mu] \frac{dz'}{\mu}, \end{aligned} \quad (44)$$

and

$$\begin{aligned} I_1(s, z, -\mu; \mu') &= \int_{z_0}^{z} \sigma(z')\bar{J}_1(s, z'; \mu') \exp[-\{q(s, z) - q(s, z')\}/\mu] \frac{dz'}{\mu}. \end{aligned} \quad (45)$$

The modified Milne's integral equations for $\bar{J}_0(s, z; \mu')$ and $\bar{J}_1(s, z; \mu')$ are

$$(1 - L)_z\{\bar{J}_0(s, z; \mu')\} = \tfrac{1}{2} \exp[-\{q(s, z) - q(s, z_0)\}/\mu'], \quad (46)$$

and

$$(1 - L)_z\{\bar{J}_1(s, z; \mu')\} = \tfrac{1}{2} \exp[-\{q(s, z_1) - q(s, z)\}/\mu']. \quad (47)$$

THEOREM 2 (Duhamel's Principle). *Let* $I(t, z, \mu)$ *be the solution of equation of transfer* (6) *subject to the initial and boundary conditions given by Eqs* (7) *through* (9). *If* $I_{\text{inc}}(t, \mu)$ *and* $I^*_{\text{inc}}(t, \mu)$ *are functions whose Laplace integrals converge for* Re $s > 0$ *and they are summable with respect to* μ *for* $0 < \mu \leqslant 1$, *then we have*

$$\begin{aligned} I(t, z, \mu) &= \int_0^t \int_0^1 I_0(t - t', z, \mu; \mu') I_{\text{inc}}(t', \mu')\, dt'\, d\mu' \\ &\quad + \int_0^t \int_0^1 I_1(t - t', z, \mu; \mu') I^*_{\text{inc}}(t', \mu')\, dt'\, d\mu'. \end{aligned} \quad (48)$$

For a semi-infinite atmosphere the second term on the right-hand side of Eq. (48) disappears.

PROOF. By the assumption of this theorem and Eq. (24), we have

$$
\begin{aligned}
|B(s, z)| &\leq \int_0^1 |I_{\mathrm{inc}}(s, \mu')| \cdot |\exp[-\{q(s, z) - q(s, z_0)\}/\mu]| \, d\mu' \\
&\quad + \int_0^1 |I^*_{\mathrm{inc}}(s, \mu')| \cdot |\exp[-\{q(s, z_1) - q(s, z)\}/\mu']| \, d\mu' \\
&= \int_0^1 |I_{\mathrm{inc}}(s, \mu')| \cdot \exp[-\{\mathrm{Re}[q(s, z) - q(s, z_0)]\}/\mu'] \, d\mu' \\
&\quad + \int_0^1 |I^*_{\mathrm{inc}}(s, \mu')| \cdot \exp[-\{\mathrm{Re}[q(s, z_1) - q(s, z)]\}/\mu'] \, d\mu' \\
&\leq \int_0^1 |I_{\mathrm{inc}}(s, \mu')| \, d\mu' + \int_0^1 |I^*_{\mathrm{inc}}(s, \mu')| \, d\mu' \leq K,
\end{aligned}
$$

where K is a non-negative constant. Hence $B(s, z)$ satisfies the assumption of Theorem 1, whence there is the unique solution of Eq. (21). Likewise, Eqs. (46) and (47) have respectively unique solutions.

Multiplying Eqs. (46) and (47), respectively, by $I_{\mathrm{inc}}(s, \mu')$ and $I^*_{\mathrm{inc}}(s, \mu')$, and integrating with respect to μ' over (0, 1), we get

$$
\begin{aligned}
(1 - L)_z &\left\{\int_0^1 I_{\mathrm{inc}}(s, \mu') \bar{J}_0(s, z'; \mu') \, d\mu'\right\} \\
&= \frac{1}{2} \int_0^1 I_{\mathrm{inc}}(s, \mu') \exp[-\{q(s, z) - q(s, z_0)\}/\mu'] \, d\mu',
\end{aligned}
$$

and

$$
\begin{aligned}
(1 - L)_z &\left\{\int_0^1 I^*_{\mathrm{inc}}(s, \mu') \bar{J}_1(s, z'; \mu') \, d\mu'\right\} \\
&= \frac{1}{2} \int_0^1 I^*_{\mathrm{inc}}(s, \mu') \exp[-\{q(s, z_1) - q(s, z)\}/\mu'] \, d\mu'.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
(1 - L)_z &\left\{\int_0^1 I_{\mathrm{inc}}(s, \mu') \bar{J}_0(s, z'; \mu') \, d\mu'\right. \\
&\left. + \int_0^1 I^*_{\mathrm{inc}}(s, \mu') \bar{J}_1(s, z'; \mu') \, d\mu'\right\} = B(s, z), \qquad (49)
\end{aligned}
$$

where $B(s, z)$ is given by Eq. (24).

Equations (21) and (49) are identical. Since these equations have respectively unique solutions, these solutions should be equal. Hence we get

$$
\begin{aligned}
\bar{J}(s, z) &= \int_0^1 I_{\mathrm{inc}}(s, \mu') \bar{J}_0(s, z; \mu') \, d\mu' \\
&\quad + \int_0^1 I^*_{\mathrm{inc}}(s, \mu') \bar{J}_1(s, z; \mu') \, d\mu'. \qquad (50)
\end{aligned}
$$

Substituting Eq. (50) into (17), we have

$$\bar{I}(s, z, +\mu) = \bar{I}^*_{\text{inc}}(s, \mu) \exp[-\{q(s, z_1) - q(s, z)\}/\mu]$$
$$+ \int_0^1 \bar{I}_{\text{inc}}(s, \mu')\, d\mu' \int_z^{z_1} \sigma(z') \bar{J}_0(s, z', \mu') \exp[-\{q(s, z') - q(s, z)\}/\mu] \frac{dz}{\mu}$$
$$+ \int_0^1 \bar{I}^*_{\text{inc}}(s, \mu')\, d\mu' \int_z^{z_1} \sigma(z') \bar{J}_1(s, z'; \mu') \exp[-\{q(s, z') - q(s, z)\}/\mu] \frac{dz'}{\mu}. \tag{51}$$

Using Eqs. (42) and (44), we obtain

$$\bar{I}(s, z, +\mu) = \int_0^1 \bar{I}_0(s, z, +\mu; \mu') \bar{I}_{\text{inc}}(s, \mu')\, d\mu' + \int_0^1 \bar{I}_1(s, z, +\mu; \mu') \bar{I}^*_{\text{inc}}(s, \mu')\, d\mu'. \tag{52}$$

On substituting Eq. (50) into (18), by Eqs. (43) and (45), we find

$$\bar{I}(s, z, -\mu) = \int_0^1 \bar{I}_0(s, z, -\mu; \mu') \bar{I}_{\text{inc}}(s, \mu')\, d\mu' + \int_0^1 \bar{I}_1(s, z, -\mu; \mu') \bar{I}^*_{\text{inc}}(s, \mu')\, d\mu'. \tag{53}$$

Combining Eqs. (56) and (57), we get

$$\bar{I}(s, z, \mu) = \int_0^1 \bar{I}_0(s, z, \mu; \mu') \bar{I}_{\text{inc}}(s, \mu')\, d\mu' + \int_0^1 \bar{I}_1(s, z, \mu; \mu') \bar{I}^*_{\text{inc}}(s, \mu')\, d\mu', \qquad (0 < |\mu| \leqslant 1). \tag{54}$$

Taking the Laplace inversion of Eq. (54) and using convolution theorem (See Widder [2]), we get Eq. (48). This completes the proof.

4.2 CASE 2. $F(t) = H(t)$.

In order to distinguish the quantities in this case from those in Case 1, we shall put a dagger to all quantities reffering to Case 2.

The initial and boundary conditions are

$$I_0^\dagger(0, z, \mu; \mu') = I_1^\dagger(0, z, \mu; \mu') = 0 \qquad (0 < |\mu| \leqslant 1), \tag{55}$$

$$I_0^\dagger(t, z_0, -\mu; \mu') = I_1^\dagger(t, z_1, +\mu; \mu') = H(t)\, \delta(\mu - \mu') \qquad (0 < \mu, \mu' \leqslant 1), \tag{56}$$

$$I_1^\dagger(t, z_0, -\mu; \mu') = I_0^\dagger(t, z_1, +\mu; \mu') = 0 \qquad (0 < \mu, \mu' \leqslant 1). \tag{57}$$

The Laplace transformations of corresponding source functions

$$\bar{J}_0^{\dagger}(s, z; \mu') \quad \text{and} \quad \bar{J}_1^{\dagger}(s, z; \mu')$$

satisfy following modified Milne's integral equations

$$(1 - L)_z\{\bar{J}_0^{\dagger}(s, z; \mu')\} = \frac{1}{2s} \exp[-\{q(s, z) - q(s, z_0)\}/\mu'], \tag{58}$$

and

$$(1 - L)_z\{\bar{J}_1^{\dagger}(s, z; \mu')\} = \frac{1}{2s} \exp[-\{q(s, z_1) - q(s, z)\}/\mu']. \tag{59}$$

Using Theorem 1 and Eqs. (21), (58), and (59), we get

$$\bar{J}(s, z) = s \int_0^1 \bar{J}_0^{\dagger}(s, z; \mu')\bar{I}_{\text{inc}}(s, \mu')\, d\mu' + s \int_0^1 \bar{J}_1^{\dagger}(s, z; \mu')\bar{I}_{\text{inc}}^{*}(s, \mu')\, d\mu'. \tag{60}$$

Substituting Eq. (60) into Eqs. (17) and (18), we have

$$\begin{aligned} \bar{I}(s, z, \mu) = s &\int_0^1 \bar{I}_0^{\dagger}(s, z, \mu; \mu')\bar{I}_{\text{inc}}(s, \mu')\, d\mu' \\ &+ s \int_0^1 \bar{I}_1^{\dagger}(s, z, \mu; \mu')\bar{I}_{\text{inc}}^{*}(s, \mu')\, d\mu'. \end{aligned} \tag{61}$$

Taking the Laplace inversion of Eq. (61), we obtain

THEOREM 3. *Under the conditions in Theorem 2, we have*

$$\begin{aligned} I(t, z, \mu) = &\int_0^t \int_0^1 \frac{\partial}{\partial t} I_0^{\dagger}(t - t', z, \mu; \mu') I_{\text{inc}}(t', \mu')\, dt'\, d\mu' \\ &+ \int_0^t \int_0^1 \frac{\partial}{\partial t} I_1^{\dagger}(t - t', z, \mu; \mu') I_{\text{inc}}^{*}(t', \mu')\, dt'\, d\mu', \end{aligned} \tag{62}$$

and for a semi-infinite atmosphere the second term on the right-hand side of Eq. (62) disappears.

5. CONCLUDING REMARKS

For the internal radiation field due to nonstationary incident radiation, Duhamel's principle is stated in the form of Theorem 2 or 3. In these expressions the radiation fields $I_0(t - t', z, \mu; \mu')$ and $I_1(t - t', z, \mu; \mu')$ are regarded as those at time t at depth z in direction $\cos^{-1} \mu$, due to incident radiation at time t' in direction $\cos^{-1} \mu'$. We shall show this fact mathematically.

Denoting these radiation fields by $I_0(t, z, \mu; t', \mu')$ and $I_1(t, z, \mu; t', \mu')$ together with initial and boundary conditions

$$I_0(t, z, \mu; t', \mu') = 0 \qquad \text{for} \qquad 0 \leqslant t \leqslant t', \tag{63}$$

$$I_0(t, z_0, -\mu; t', \mu') = \delta(t - t')\, \delta(\mu - \mu'), \tag{64}$$

$$I_0(t, z_1, +\mu; t', \mu') = 0, \tag{65}$$

and

$$I_1(t, z, \mu; t', \mu') = 0 \qquad \text{for} \qquad 0 \leqslant t \leqslant t', \tag{66}$$

$$I_1(t, z_0, -\mu; t', \mu') = 0, \tag{67}$$

$$I_1(t, z_1, +\mu; t', \mu') = \delta(t - t')\, \delta(\mu - \mu'). \tag{68}$$

and corresponding source functions by $J_0(t, z; t', \mu')$ and $J_1(t, z; t', \mu')$, we get

$$(1 - L)_z \{\bar{J}_0(s, z'; t', \mu')\} = \tfrac{1}{2}\bar{e}^{st'} \exp[-\{q(s, z) - q(s, z_0)\}/\mu']$$

and

$$(1 - L)_z \{\bar{J}_1(s, z'; t', \mu')\} = \tfrac{1}{2}\bar{e}^{st'} \exp[-\{q(s, z_1) - q(s, z)\}/\mu'].$$

By Eqs. (46) and (47), we have

$$\bar{J}_0(s, z; t', \mu') = \exp(-st')\, \bar{J}_0(s, z; \mu'), \tag{69}$$

and

$$\bar{J}_1(s, z; t', \mu') = \exp(-st')\, \bar{J}_1(s, z; \mu'). \tag{70}$$

The Laplace transformations of Eqs. (64), (65), (67) and (68) give

$$\begin{aligned} &\bar{I}_0(s, z_0, -\mu; t', \mu') \\ &\qquad = \bar{I}_1(s, z_1, +\mu; t', \mu') = \exp(-st')\, \delta(\mu - \mu'), \end{aligned} \tag{71}$$

$$\bar{I}_0(s, z_1, +\mu; t', \mu') = \bar{I}_1(s, z_0, +\mu; t', \mu') = 0. \tag{72}$$

Inserting Eqs. (69) through (72) into Eqs. (17) and (18), and using Eqs. (42) through (45), we have

$$\bar{I}_0(s, z, \mu; t', \mu') = \exp(-st')\, \bar{I}_0(s, z, \mu; \mu'), \tag{73}$$

and

$$\bar{I}_1(s, z, \mu; t', \mu') = \exp(-st')\, \bar{I}_1(s, z, \mu; \mu'). \tag{74}$$

Taking the Laplace inversion of Eqs. (73) and (74), we get

$$I_0(t, z, \mu; t', \mu') = I_0(t - t', z, \mu; \mu'). \tag{75}$$

and

$$I_1(t, z, \mu; t', \mu') = I_1(t - t', z, \mu; \mu'). \tag{76}$$

This fact may be inferred from physical aspect.

While for the stationary radiation field Duhamel's principles for a finite and semi-infinite atmosphere hold respectively under the conditions σ/ℓ [illegible] 1 and σ/ℓ [illegible] 1, for the nonstationary radiation field this difference disappears, because Theorem 1 is valid even if $z_1 = \infty$.

Acknowledgment

The author is deeply indebted to Professor S. Ueno for his kind suggestion and advice. He also expresses his gratitude to Professor R. Bellman for the helpful comment on this work.

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